Is $\sigma_A^2 = \sigma_B^2$?

If $\sigma_A^2 \neq \sigma_B^2$ we have

$$T = \frac{\bar{Y} - \bar{X} - (\mu_B - \mu_A)}{\sqrt{\frac{S_A^2}{10} + \frac{S_B^2}{10}}} \approx t\text{-distributed with}$$

$$V = \frac{\left(\frac{S_A^2}{10} + \frac{S_B^2}{10}\right)^2}{\left(\frac{S_A^2}{10}\right)^2/9 + \left(\frac{S_B^2}{10}\right)^2/9}$$

$$t_{obs} = \frac{85.54 - 84.24}{\sqrt{\frac{(2.9)^2}{10} + \frac{(3.65)^2}{10}}} = 0.88$$

$V = 17 \Rightarrow P(T \geq t_{obs} \mid H_0) = 0.195$

An alternative approach

Use of historical data

For method A we have 210 data: $x_1, \dots, x_{210}$

1: A: $x_1, \dots, x_{10}$ B: $x_{11}, \dots, x_{20}$ $(\bar{X}_B - \bar{X}_A)_1$

2: A: $x_2, \dots, x_{11}$ B: $x_{12}, \dots, x_{21}$ $(\bar{X}_B - \bar{X}_A)_2$

⋮

191: A: $x_{191}, \dots, x_{200}$ B: $x_{201}, \dots, x_{210}$ $(\bar{X}_B - \bar{X}_A)_{191}$

Only 9 out of 191 differences are greater than 1.3

$P(\bar{X}_B - \bar{X}_A > 1.3) \approx \frac{9}{191} = 0.047 < 0.05$ and may indicate that Method B is better than Method A

How to get independent data that can be used with the random sample approach

Used method: $\underbrace{AA \text{---} A}\ \underbrace{BB \text{--} B}$

Recommended method: A A B A B B A - - - -

i.e randomize the order in which the methods are tried out

By making referencedistributions based on historical data, the problem with correlated data will be avoided.

Problem. Historical data are not necessarily representative.

The t- observator is robust against deviations from the normal distribution, but not robust against correlation in the data

The use of t-observator

One sample, σ^2 unknown

$X_1, \ldots, X_n \sim N(\mu, \sigma^2)$ and independent. $T = \dfrac{\bar{X} - \mu}{\frac{S}{\sqrt{n}}}$ is

t-distributed with $(n-1)$ degrees of freedom.

$S = \sqrt{\dfrac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2}$, Chap (8)

Hypothesis tests: $H_0: \mu = \mu_0$ $\quad H_1: \mu \overset{>}{\underset{<}{\neq}} \mu_0$, Chap (10)

Confidence interval: $\bar{X} \pm t_{\frac{\alpha}{2}, n-1} \dfrac{S}{\sqrt{n}}$

Two samples $\sigma_1^2 = \sigma_2^2$

$$\left.\begin{array}{l} X_1, \ldots, X_n \sim N(\mu_1, \sigma^2) \\ Y_1, \ldots, Y_m \sim N(\mu_2, \sigma^2) \end{array}\right\} \text{independent.} \quad T = \frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{S\sqrt{\frac{1}{n} + \frac{1}{m}}} \quad \text{Chap (9)}$$

is t-distributed with $n+m-2$ degrees of freedom.

$$S = \sqrt{\frac{\sum_{i=1}^{n} (X_i - \bar{X})^2 + \sum_{j=1}^{m} (Y_j - \bar{Y})^2}{n + m - 2}}$$

Hypothesis test: $H_0: \mu_1 - \mu_2 = 0$ $\quad H_1: \mu_1 \overset{>}{\underset{<}{\neq}} \mu_2$ Chap 10

Confidence interval: $\bar{X} - \bar{Y} \pm t_{\frac{\alpha}{2}, n+m-2}\, S \sqrt{\frac{1}{n} + \frac{1}{m}}$ (Chap 9).

Paired t-test

$D_i = X_i - Y_i \sim N(\delta, \sigma^2)$, $i = 1, 2, \ldots, n$ and independent

$T = \dfrac{\bar{D} - \delta}{\frac{S_D}{\sqrt{n}}} \sim$ t-distributed with $(n-1)$ degrees of freedom

$S_D = \sqrt{\dfrac{1}{n-1} \sum_{i=1}^{n} (D_i - \bar{D})^2}$

Hypothesis test : $H_0: \delta = \delta_0$ $\quad H_1: \delta \overset{>}{\underset{<}{\neq}} \delta_0$ Chap 10

Confidence interval : $\bar{d} \pm t_{\frac{\alpha}{2}, n-1} \frac{s_D}{\sqrt{n}}$ Chap 9

10.10 Test for the inequality of two variances

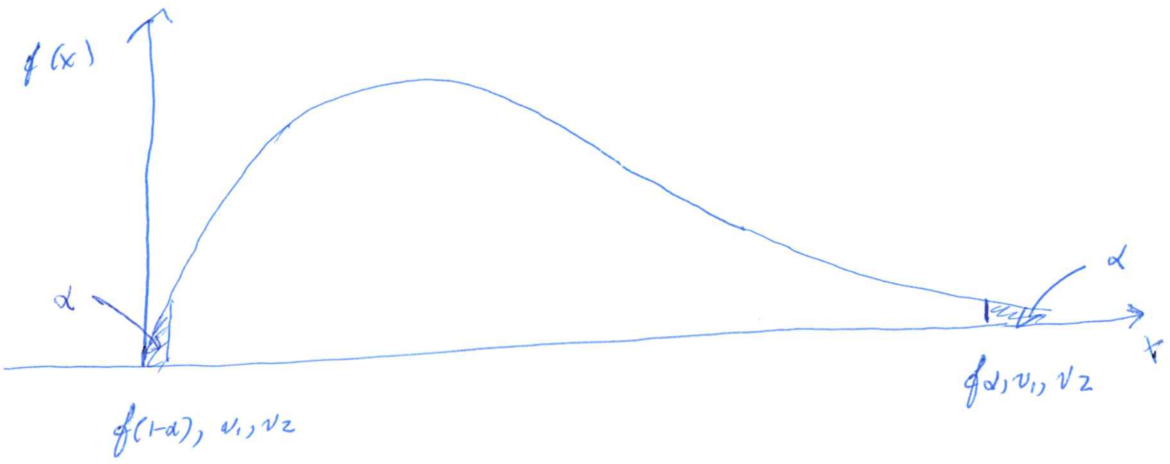
Let $X_i \sim N(\mu, \sigma^2)$ and independent, $i = 1, 2, \dots, n$

Estimator for σ^2: $S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$

and $(n-1)\frac{S^2}{\sigma^2}$ is $\chi^2(n-1)$ (Theorem 8.4, Theorem 8.6)

Theorem 8.6. Let U be $\chi^2(v_1)$ and V be $\chi^2(v_2)$ and independent. Then $F = \frac{U/v_1}{V/v_2}$ is Fisher distributed with v_1 and v_2 degrees of freedom.

There are table-values for the F-distribution, but only for f_{α, v_1, v_2}.



How to find $f_{(1-\alpha), v_1, v_2}$?

We have $P(F \geq f_{\alpha, v_1, v_2}) = P\left(\frac{U/v_1}{V/v_2} \geq f_{\alpha, v_1, v_2}\right)$

$= P\left(\frac{V/v_2}{U/v_1} \leq \frac{1}{f_{\alpha, v_1, v_2}}\right) \Rightarrow \frac{1}{f_{\alpha, v_1, v_2}} = f_{1-\alpha, v_2, v_1}$

such that $f_{1-\alpha, v_1, v_2} = \frac{1}{f_{\alpha, v_2, v_1}}$

Let X_{1i} be independent $N(\mu_1, \sigma_1^2)$, $i = 1, 2, \dots, n_1$

and X_{2i} — " — $N(\mu_2, \sigma_2^2)$, $i = 1, 2, \dots, n_2$

and independent of X_{1i}, $i = 1, 2, \dots, n_1$

Then $\frac{n_1 - 1}{\sigma_1^2} S_1^2 = \sum_{i=1}^{n_1} \left(\frac{X_{1i} - \bar{X}_1}{\sigma_1}\right)^2 \sim \chi^2(n_1 - 1)$

and $\frac{n_2 - 1}{\sigma_2^2} S_2^2 = \sum_{i=1}^{n_2} \left(\frac{X_{2i} - \bar{X}_2}{\sigma_2}\right)^2 \sim \chi^2(n_2 - 1)$

We get $\dfrac{\frac{S_1^2}{\sigma_1^2}}{\frac{S_2^2}{\sigma_2^2}} \sim$ F- distributed with (n_1-1) and (n_2-1) degrees of freedom.

$H_0: \sigma_1^2 = \sigma_2^2$ $\qquad H_1: \sigma_1^2 \neq \sigma_2^2$

Under H_0, $\frac{S_1^2}{S_2^2}$ is F distributed with $n_1 - 1$ and $n_2 - 1$ degrees of freedom. Reject H_0 if $\frac{S_1^2}{S_2^2} > f_{\frac{\alpha}{2}, n_1-1, n_2-1}$

or $\frac{S_1^2}{S_2^2} < \frac{1}{f_{\frac{\alpha}{2}, n_2-1, n_1-1}}$

The industrial example.

$H_0: \sigma_A^2 = \sigma_B^2$ $\qquad H_1: \sigma_A^2 \neq \sigma_B^2$

$\frac{S_A^2}{S_B^2} = \frac{(2.902)^2}{(3.65)^2} = 0.63$

~~$f_{0.975, 9, 9} = 4.03$~~

$f_{0.025, 9, 9} = 4.03$, $\qquad f_{0.975, 9, 9} = \frac{1}{4.03} \approx 0.248$

Since $0.248 < 0.6 < 4.03$, H_0 is not rejected on a 5% significance level